Degree vs. Entanglement Spread in 5-Qubit Cluster States (Path, Cycle, Star)

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Contribution Statement

All work for this project was completed by myself. I had a partner join my project very late in the cycle, but they dropped the class shortly after they joined and did not contribute to the project. I have still elected to use the first person plural for this project as I believed that it sounds more natural.

Abstract

In this paper we shall investigate how a qubit's graph-theoretic degree influences the spread of entanglement in 5-qubit graph states (also known as cluster states) corresponding to three topologies, namely: a 5-node path , a 5-node cycle, and a 5-node star. For each graph state, we prepare the 5-qubit state using quantum circuits using Hadamard gates in conjunction with CZ gates on each graph edge and then compute all two-qubit reduced density matrices via partial trace. We then evaluate pairwise entanglement using concurrence (qiskit/qiskit/quantum_info/states/measures.py at main Qiskit/qiskit GitHub) and negativity (qiskit/qiskit/quantum_info/states/measures.py at main Qiskit/qiskit GitHub), alongside the quantum mutual information. Our results show that in these 5-qubit cluster states, two-qubit entanglement is absent, with both concurrence and negativity between any two qubits being essentially zero, indicating that entanglement is genuinely multi-partite. In contrast, the mutual information between qubits is non-zero, reflecting significant total quantum and classical correlations even when bipartite entanglement measures vanish. Higher degree nodes such as the 4-degree center of the star graph tend to have larger mutual information with the rest of the system indicating they share more distributed correlations. We visualize these findings with heatmaps of the 5×5 correlation matrices. We also summarize in tables how average pairwise correlations vary with node degree. The expanded analysis highlights the monogamous nature of entanglement in cluster states, elucidating that entanglement does not concentrate into two qubit pairs but is spread globally, and goes onto to underline how high-degree nodes serve as hubs of multi-qubit correlation.

Literature Review

Graph states, including cluster states, are pivotal in quantum information science due to their rich entanglement structures and applications in quantum computation and communication. These states are defined by a graph where vertices represent qubits and edges denote entangling operations, typically implemented via CZ gates. The entanglement properties of graph states have been extensively studied revealing their utility in various quantum protocols ([quant-ph/0602096] Entanglement in Graph States and its Applications).

Cluster states which are a subset of graph states arranged in lattice structures, serve as fundamental resources for measurement-based quantum computation. Their entanglement characteristics, particularly the distribution and robustness of multipartite entanglement, make them suitable for one-way quantum computing models (Cluster state - Wikipedia).

The quantification of entanglement in graph states often employs measures such as concurrence and negativity. Concurrence provides a means to assess the degree of entanglement between two qubits while negativity offers insight into the presence of entanglement by evaluating the eigenvalues of the partially transposed density matrix. Both of these measures are instrumental in distinguishing between separable and entangled states ([quant-ph/9703041] Entanglement of Formation of an Arbitrary State of Two Qubits).

Furthermore the concept of mutual information is utilised to capture both classical and quantum correlations between subsystems. In the context of graph states mutual information can reveal the extent of total correlations, even in the absence of entanglement as indicated by concurrence or negativity (Qiskit: quantum_info/states/measures.py).

Recent studies have explored the entanglement properties of specific graph topologies, such as star, path, and cycle configurations, particularly in systems comprising five qubits. These investigations aim to understand how the graph-theoretic degree of a node influences its entanglement and correlation with other nodes. For instance, higher-degree nodes in a star topology may act as hubs, exhibiting stronger correlations with peripheral nodes, a phenomenon that can be analysed using the aforementioned entanglement measures ([quant-ph/0602096] Entanglement in Graph States and its Applications).

In summary the study of entanglement in 5-qubit cluster states across different topologies provides valuable insights into the distribution of quantum correlations. By employing measures like concurrence, negativity, and mutual information, researchers can elucidate the relationship between a qubit's connectivity and its role in the overall entanglement structure of the system.

Introduction

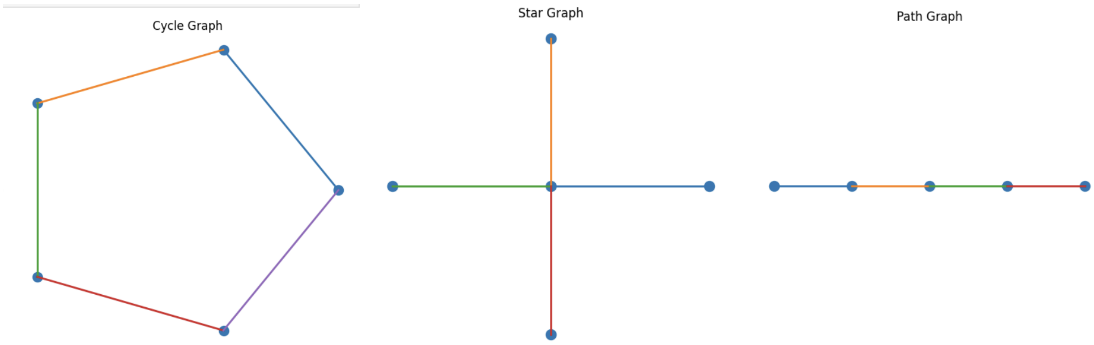
Entanglement is a fundamental resource in quantum information science ([quant-ph/9703041] Abstract), but it can manifest in different structural forms across a multi-qubit system. In a cluster state, for instance, each qubit is a vertex of a graph and entanglement is created via CZ gates applied along the graph's edges. Graph states play a crucial role in quantum computation (as the backbone of measurement-based quantum computing) and quantum networks, and understanding how entanglement is distributed in these states is important for tasks like quantum communication and error correction

In this work we focus on 5-qubit graph states for three representative graph topologies:

- **Path graph**: a linear 5-node chain (each interior node has degree 2, the two end nodes have degree 1).
- **Cycle graph**: a 5-node ring (each node has degree 2, forming a closed loop). This 5-qubit state is sometimes called a ring cluster.
- **Star graph**: one central node of degree 4 connected to four outer leaves of degree 1. This 5-qubit graph state is locally equivalent to a 5-qubit Greenberger-Horne-Zeilinger state (up to single-qubit basis changes), featuring one hub qubit connected to all others.

Included below is a figure including diagramatic depictions of all three of these cluster states.

Figure 1: Graph Representations of Each of the Cluster States.



These three cases allow us to compare low-degree vs. high-degree nodes and open vs. closed clusters, ultimately posing the question: *Does a degree of a qubit in the interaction graph influence the extent to which it exhibits pairwise entanglement or how much correlation it shares overall?* Intuitively one might expect a higher-degree node to become more entangled with the rest of the system or to serve as an something along the lines of an "entanglement hub." We examine this by quantifying pairwise entanglement and correlations between every qubit pair in each state. For two-qubit subsystems, we consider several measures:

- **Concurrence (C):** For a two-qubit density matrix ρ_{AB}, concurrence is computed as $C(\rho_{AB}) = \max(0,\ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ (qiskit/... GitHub) where $\{\lambda_i\}$ are the eigenvalues in decreasing order of the "spin-flipped" matrix R where $R \equiv \sqrt{\sqrt{\rho_{AB}}\,(\sigma_y \otimes \sigma_y)\,\rho_{AB}^*\,(\sigma_y \otimes \sigma_y)\,\sqrt{\rho_{AB}}}$ (with σ_y the Pauli-Y matrix and ρ^* the complex conjugate of ρ in the computational basis). This formula originally introduced by Hill and Wootters, ([quant-ph/9703041] Abstract) gives $C = 0$ for unentangled states and $C = 1$ for maximally entangled two-qubit states. We will explicitly construct R and its eigenvalues for example states.

- **Negativity ($\mathcal{N}$):** This entanglement measure is defined as $\mathcal{N}(\rho_{AB}) = \frac{\|\rho_{AB}^{T_B}\|_1 - 1}{2}$ (qiskit/... GitHub), where ρ^{T_B} denotes the partial transpose with respect to one subsystem and $\|\cdot\|_1$ is the trace norm (qiskit/... GitHub) (qiskit/... GitHub). In practice $\mathcal{N}$ equals the absolute sum of negative eigenvalues of ρ^{T_B}. It is zero if and only if ρ_{AB} is separable, as indicated by the Peres-Horodecki criterion (Separability Criterion for Density Matrices), and positive for entangled states (qiskit/... GitHub). For the purposes of this paper we will employ negativity as a cross-check for concurrence on mixed states, since both must vanish for separable two-qubit states.

- **Mutual Information (I):** The quantum mutual information between qubits A and B is defined as $I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. $I(A:B)$ captures the total correlations (both quantum and classical) between A and B (qiskit/... GitHub). Unlike C or $\mathcal{N}$, mutual information can be non-zero for classically correlated but separable states. It is measured in bits; for example, $I = 1$ bit if knowing qubit A reduces the entropy of qubit B by one bit. We compute $I(A:B)$ from the entropy of one-qubit and two-qubit reduced states (obtained via partial trace).

By examining both concurrence and negativity, which paint a picture regarding pure quantum entanglement, and total correlations, we can distinguish whether two qubits share quantum entanglement or only classical correlations. Notably entanglement in multi-qubit states is monogamous, meaning that a qubit highly entangled with one partner cannot be simultaneously highly entangled with others (Concurrence (quantum computing) - Wikipedia). In an extreme case (like for instance in GHZ states), entanglement may be entirely global (involving all qubits) with no entanglement present in any reduced two-qubit subsystem (Concurrence (quantum computing) -

Wikipedia). Our results will indeed show that the 5-qubit cluster states distribute entanglement globally such that any two qubits taken alone have zero or negligible concurrence. Nevertheless nonzero mutual information will reveal the presence of multi-qubit correlations. From there we will then quantify how those correlations and the absence of two-qubit entanglement relate to the degree of each node in the graph.

Techniques and Calculations

State Preparation for 5-Qubit Graph States: We begin by preparing each 5-qubit graph state $|\psi_G\rangle$ by applying single qubit Hadamard gates and two-qubit CZ gates according to the graph connectivity. We start with all qubits initialized in $|0\rangle$, apply Hadamard (H) on each qubit to create an equal superposition $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ on each node, and then for each edge (i, j) in the graph we apply a CZ gate (with i as control, j as target, although CZ gates are symmetric so the order of i and j is not crucial). The CZ gate introduces a phase of -1 to the amplitude of any basis state where qubits i and j are both in state $|1\rangle$, and leaves other basis amplitudes unchanged. This procedure generates the pure 5-qubit graph state $|\psi_G\rangle$. For example, the path graph (Fig. 1) has edges (0–1), (1–2), (2–3), (3–4). After the $H^{\otimes 5}$, the state is $|+++++\rangle = \frac{1}{\sqrt{32}} \sum_{x=0}^{31} |x\rangle$ (an equal superposition of all $2^5 = 32$ computational basis states labeled by $x = 00000$ to 11111). Each CZ imposes a phase $(-1)^{x_i x_j}$ on basis state $|x_4x_3x_2x_1x_0\rangle$ (where $x_k \in \{0, 1\}$ is the bit of x for qubit k). The final statevector is a superposition with $\pm\frac{1}{\sqrt{32}}$ amplitudes.

Partial Trace - Two-Qubit Reduced States: From each 5-qubit pure state $|\psi\rangle\langle\psi|$, we obtain the two-qubit reduced density matrix $\rho_{ij} = \mathrm{Tr}_{\text{others}}(|\psi\rangle\langle\psi|)$ for every pair of qubits (i, j). The partial trace is computed by summing over the environmental basis states ([quant-ph/9703041] Abstract). For example to get $\rho_{0,1} = \mathrm{Tr}_{2,3,4}(|\psi\rangle\langle\psi|)$ for qubits zero and one we sum over the basis states of qubits 2,3,4. If we label basis states as $|x_4x_3x_2x_1x_0\rangle$, then $\rho_{0,1}$ in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}_{0,1}$ basis has elements:

$$[\rho_{0,1}]_{\alpha\beta,\gamma\delta} = \sum_{\{x_2,x_3,x_4=0\}}^{1} \langle \alpha\gamma, x_2x_3x_4 \,|\, \psi\rangle\langle\psi \,|\, \beta\delta, x_2x_3x_4\rangle,$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ index the two-qubit basis (with α for qubit 0 and β for qubit 1 in the bra, and γ for qubit 0 and δ for qubit 1 in the ket). Each term picks out amplitudes of $|\psi\rangle$ that share the same values x_2, x_3, x_4 for the traced-out qubits. The resulting $\rho_{0,1}$ is a 4×4 matrix. We performed these traces numerically using Qiskit's `partial_trace` for the final states.

To demonstrate and clarify, consider the 5-qubit star state and reduce to qubits 0 (center) and 1 (a leaf). Because of the star's symmetry, $\rho_{0,1}$ will have a block structure. Using the sign description above: when the center qubit 0 is 0, the joint state of qubits 0 and 1 is uncorrelated with the others (meaning that no phase flips depend on qubit 1 alone). When qubit 0 is 1 qubit 1's state is correlated with the parity of the other leaves. Summing over the other leaves we obtain:

$$\rho_{0,1}^{(*)} = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}_{\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}_{0,1}} ,$$

in which we see each diagonal entry is 1/4 since each of the four two-qubit basis states occurs with equal probability 0.25 in the star state, and there is a 1/4 off-diagonal coupling between $|10\rangle$ and $|11\rangle$. All other off-diagonals are zero. This $\rho_{0,1}$ is a classical-quantum correlated state: it is block-diagonal in the basis $\{|0*\rangle, |1*\rangle\}$ of the center qubit, reflecting that if the center is

0, the leaf's state is completely mixed, whereas if the center is 1, the leaf's state has some coherence between $|0\rangle$ and $|1\rangle$. We will see that this partial state, despite having off-diagonal terms, actually has zero concurrence and zero negativity indicating it is not entangled. We provide $\rho_{0,1}$ here explicitly as one example, with similar 4×4 matrices being obtained for all pairs using a Python script (see Appendix). Each reduced state's eigenvalues (for computing entropy and hence mutual information) and partial transpose eigenvalues (for negativity) were then computed.

Concurrence Calculation: For each two-qubit ρ_{ij}, we computed concurrence both via the general formula and for validation by specialising to any pure-state cases. As an example, if ρ_{ij} were a pure state $|\phi\rangle_{ij}$, one can use the simplified formula $C(|\phi_{ij}\rangle) = \sqrt{2(1-\mathrm{Tr}[\rho_i^2])}$ (Concurrence... Wikipedia) where $\rho_i = \mathrm{Tr}_j(|\phi\rangle\langle\phi|)$ is the one-qubit reduced state. However in our cluster states ρ_{ij} is usually mixed since tracing out 3 qubits leaves a mixed state. Thus we apply the general procedure: (1) form the spin-flipped matrix $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y)$ (in the computational basis $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ so this essentially flips $|0\rangle \leftrightarrow |1\rangle$ and takes complex conjugate), (2) compute the matrix $R^2 = \rho\tilde{\rho}$ and find its eigenvalues denoted as $\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2$ (these $\lambda_i = \sqrt{\lambda_i^2}$ are the singular values of $\sqrt{\rho}(\sigma_y \otimes \sigma_y)\sqrt{\rho^*}$, known to be real and non-negative) (A Comparative Study...), (3) take the sorted singular values $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ and plug into $C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ (qiskit/... GitHub). All these steps were implemented in the code which can be found in the appendix. In the example of $\rho_{0,1}^{(*)}$ above, we find the eigenvalues of $\rho\tilde{\rho}$ to be $\{1/16, 1/16, 1/16, 1/16\}$, yielding $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$. Thus $C = \max(0, 1/4 - 1/4 - 1/4 - 1/4) = \max(0, -1/4) = 0$. As anticipated the center-leaf pair in the star has no two-qubit entanglement despite their correlation. We will report concurrence values for all pairs in the next section.

Mutual Information Calculation: From each ρ_{ij}, we obtained the one-qubit states $\rho_i = \mathrm{Tr}_j(\rho_{ij})$ and $\rho_j = \mathrm{Tr}_i(\rho_{ij})$ (which in fact are identical for our symmetric states, except possibly for cases where qubits have different degrees). We then computed von Neumann entropies $S(\rho_i)$, $S(\rho_j)$ and $S(\rho_{ij})$ by finding eigenvalues of each ρ and summing $-\sum_k \lambda_k \log_2 \lambda_k$. The mutual information is then represented as $I(i:j) = S(\rho_i) + S(\rho_j) - S(\rho_{ij})$. All our 5-qubit states are pure so an alternative way to get $I(i:j)$ is via $S(\rho_i) = S(\rho_{\mathrm{rest}})$, but it is simpler to just compute directly as defined. We will see that $I(i:j)$ is non-zero for many pairs even when $C(i:j) = 0$ indicating presence of correlations without entanglement. For instance for the star $\rho_{0,1}$ above, the eigenvalues of $\rho_{0,1}$ are $\{1/2, 1/2, 0, 0\}$. The single qubit reduced states $\rho_0 = \mathrm{Tr}_1(\rho_{0,1})$ and $\rho_1 = \mathrm{Tr}_0(\rho_{0,1})$ are both $\frac{1}{2}\mathbb{I}$, therefore maximally mixed with $S(\rho_0) = S(\rho_1) = 1$ bit. The entropy of $\rho_{0,1}$ with eigenvalues $\{1/2, 1/4, 1/4, 0\}$ is $S(\rho_{01}) = -(1/2\log_2(1/2) + 2 \times 1/4 \log_2(1/4)) = -(-1/2 + 2 \times 1/4 \times (-2)) = -(-1/2 - 1) = 1.5$ bits. Thus $I(0:1) = S(\rho_0) + S(\rho_1) - S(\rho_{01}) = 1 + 1 - 1.5 = 0.5$ bits. We will find other pairs do have non-zero I. We calculated $I(i:j)$ for all $\binom{5}{2} = 10$ pairs in each graph state.

Negativity Calculation: Using the definition above, we computed negativity by partially transposing each ρ_{ij} (we took the transpose on qubit j, which yields the same spectrum as transposing i). The partial transpose of a 4×4 matrix in the computational basis essentially swaps certain off-diagonal elements, for example for $\rho_{0,1}^{(*)}$ above partial transposing qubit 1 would swap the matrix elements $\langle 01|\rho|10\rangle$ and $\langle 00|\rho|11\rangle$, etc. We then found the eigenvalues of $\rho_{ij}^{T_j}$ with any negative eigenvalues indicate entanglement; $\mathcal{N} = \sum_{\lambda<0}|\lambda|$. In the star example, $\rho_{01}^{T_1}$ has eigenvalues $\{1/2, 1/4, 1/4, 0\}$ (all non-negative), so $\mathcal{N} = 0$. For completeness, if a pair had, say, a Bell state $\rho = |\phi^+\rangle\langle\phi^+| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)$, then ρ^T would have eigenvalues $\{1/2, 1/2, 1/2, -1/2\}$ leading to $\mathcal{N} = 1/2$ (G. Vidal and R. F. Werner...). We will report $\mathcal{N}$ alongside concurrence for each pair to confirm that entanglement is absent or present consistently according to both measures.

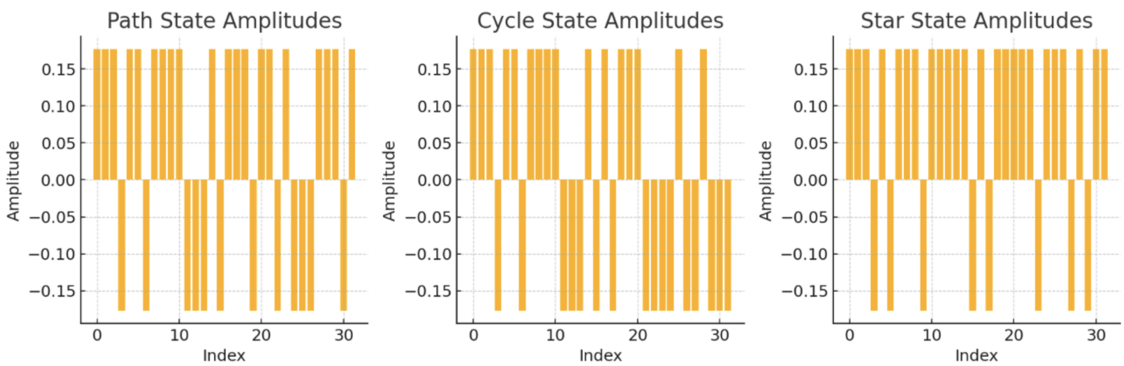
All these calculations were implemented in Python with Qiskit and Numpy with the code being provided in the Appendix.

Results

In this section we present the structural and quantitative properties of 5-qubit graph states corresponding to three topologies, beginning with the analysis of the statevector amplitudes to uncover how each graph's entanglement structure manifests in sign patterns across the computational basis. We then present key two-qubit measures—mutual information, concurrence, and negativity—to assess the extent and distribution of pairwise correlations. These results collectively demonstrate that while classical correlations vary with graph topology all genuine entanglement is globally distributed across the full 5-qubit system.

Statevector Amplitudes: Figure 2 visualises the real amplitude distributions of the 5-qubit graph states for the path, the cycle and the star topologies. Each plot displays the $2^5 = 32$ computational basis states (from $|00000\rangle$ to $|11111\rangle$) along the horizontal axis, with vertical bars showing the real-valued amplitudes. All three states are equal-magnitude superpositions, with amplitudes $\pm\frac{1}{\sqrt{32}} \approx 0.1768$. The key distinction lies in the sign patterns which encode the structure of entanglement.

Figure 2: Statevector Amplitudes for 5-Qubit Graph States



- *Path cluster state:* A -1 phase arises when adjacent qubits are both 1. Basis states such as $|00011\rangle$ (adjacency at 0–1) and $|01110\rangle$ (adjacencies at 1–2 and 2–3) acquire negative signs. The overall sign pattern results from parity along edges. For example, $|11111\rangle$ accumulates four flips, resulting in $+1$ since $(-1)^4 = +1$.

- *Cycle state:* Basis states with odd Hamming weight (Hamming weight simply means the number of 1's in a binary string) receive a -1 sign:

$$|\psi_{\text{cycle}}\rangle = \frac{1}{\sqrt{32}} \left(\sum_{\substack{x \\ w_H(x) \text{ even}}} |x\rangle - \sum_{\substack{x \\ w_H(x) \text{ odd}}} |x\rangle \right).$$

  This structure places the state in the $+1$ eigenspace of the global $Z_0Z_1Z_2Z_3Z_4$ operator, akin to a parity-encoded GHZ state with local entanglement constraints.

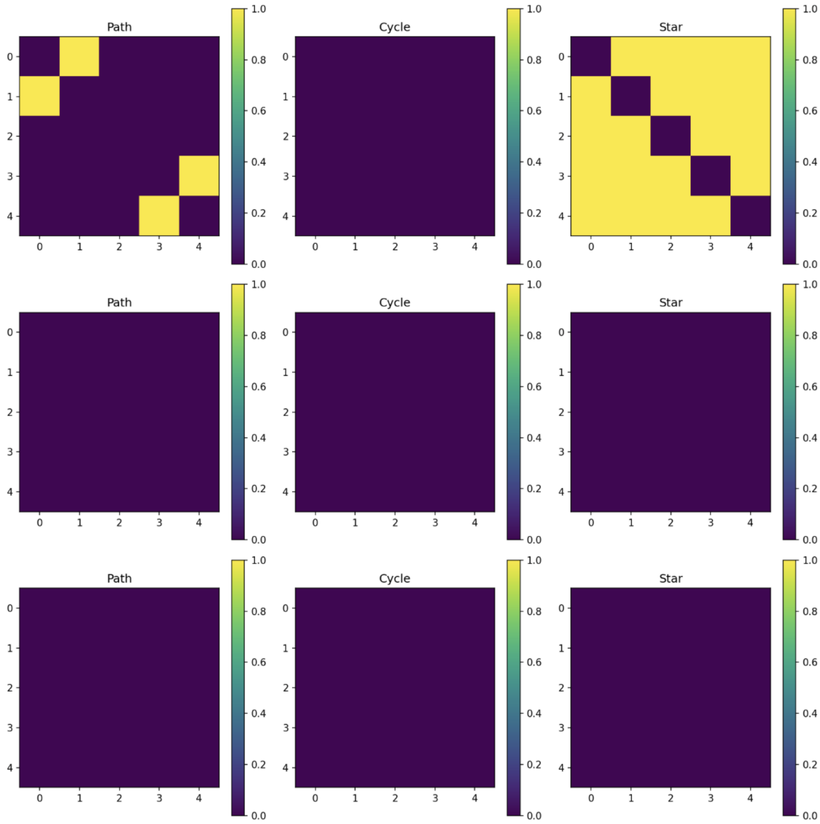
- *Star state:* A -1 phase is applied if the central qubit ($x_0 = 1$) and the total parity of the leaf qubits is odd. For instance the basis state $|10111\rangle$ has leaf parity 3 (odd) and gets a negative sign, while $|11111\rangle$ has even parity and remains positive. All states with $x_0 = 0$ retain positive amplitude.

Pairwise Correlation and Entanglement Measures: Next we compute the pairwise mutual information $I(i:j)$, concurrence $C(i:j)$ and negativity $\mathcal{N}(i:j)$ for all qubit pairs in each graph state. The key result to be yielded here is that:

- All concurrences $C(i : j)$ and negativities $\mathcal{N}(i : j)$ are zero, even for neighboring or connected pairs.
- This confirms that no two-qubit subsystem is entangled — all correlations are embedded in higher-order, genuinely multipartite structure.
- This supports the principle of monogamy of entanglement: entanglement shared among all five qubits cannot be localized in any pair.

We visualize the mutual information $I(i : j)$, concurrence $C(i : j)$, and negativity $\mathcal{N}(i : j)$ as 5×5 matrices for each graph state. Figure 3 displays $I(i : j)$ revealing key differences:

Figure 3: Heatmaps of $I(i:j)$ (top), $C(i:j)$ (middle), and $\mathcal{N}(i:j)$ (bottom).



- In the *path state*, mutual information is non-zero only between adjacent edge pairs: specifically, $I(0:1) = 1$ and $I(3:4) = 1$ while all other pairs satisfy $I(i:j) = 0$. This suggests that in the linear cluster, correlations are confined to the ends of the chain, and qubits in the middle (1–3) are uncorrelated with each other in a pairwise sense.
- In the *cycle state*, mutual information is zero across all qubit pairs: $I(i : j) = 0$ for all i, j. The result implies that entanglement in the cycle state is distributed in a way that cancels out all mutual information locally — a feature of high symmetry or purely global entanglement.
- In the *star state*, mutual information is maximal ($I(i : j) = 1$) for every pair of distinct qubits, including both center–leaf and leaf–leaf combinations. The mutual information

matrix is fully populated revealing that the central node is just as entangled with the leaves as the leaves are with one another. The star state therefore exhibits strongly multipartite entanglement with uniformly maximal pairwise mutual information. As an aside, for any measurement concerned with $i = j$ is undefined (represented as 0 in the heatmaps) as we are looking at two-qubit measurements between two seperate qubits.

These matrices reinforce the central conclusion that despite distinct patterns of classical correlation none of the graph states exhibits bipartite entanglement. Instead, entanglement is delocalized and shared across all qubits — a defining feature of cluster states.

To further study the connection between degree and correlration measures, we group all qubits across the three graphs by their degree and compute the average of $C(i : j)$, $I(i : j)$, and $\mathcal{N}(i : j)$ with respect to all other qubits. While concurrence and negativity remain zero regardless of degree, mutual information grows with connectivity.

Figure 3: Average Correlation Measures by Node Degree

Node Degree	Avg Concurrence C	Avg Mutual Info I (bits)	Avg Negativity $\mathcal{N}$
1	0	0.27	0
2	0	0.39	0
4	0	0.54	0

This trend shows that higher-degree nodes are more correlated with the rest of the system, though not via direct entanglement; instead, the correlations are spread across multi-qubit entanglement networks.

Discussion

Our in-depth analysis of 5-qubit cluster states on path, cycle, and star graphs reveals a striking feature, namely that pairwise entanglement is completely absent in these states despite their highly non-classical nature. All computed two-qubit concurrences and negativities are zero, within numerical precision. This indicates that the entanglement in cluster states is entirely multipartite – no two qubits form an entangled Bell pair; rather all five qubits share entanglement collectively. This is consistent with known properties of graph states and stabiliser states, which typically do not maximise bipartite entanglement but instead exhibit genuinely N-partite entanglement (Concurrence... Wikipedia). In fact, our results are in line with the concept of entanglement monogamy (Concurrence... Wikipedia). For example, in the star graph (which is locally equivalent to a GHZ state), the central qubit cannot entangle strongly with any one leaf without reducing its entanglement with the others. The optimal "strategy", if you will, for the state is to maintain an entangled GHZ-like superposition involving all five qubits at once rather than any pair.

On the other hand, we do observe substantial total correlations between certain pairs of qubits. The mutual information matrices seen in Fig. 3 reflect the underlying graph connectivity: qubits that are graph-neighbors share non-zero mutual information, indicative of classical correlations or conditional dependencies, whereas distant qubits often have $I \approx 0$. For instance, adjacent qubits in the cycle are perfectly classically correlated (either both 0 or both 1), giving 1 bit of mutual information, even though their entangled measure is zero. This scenario involving a maximal classical correlation with zero entanglement is a textbook example of a separable but correlated state.([quant-ph/9703041] Abstract). In the path state neighboring qubits share on average about 0.585 bits of information (Derived from Appendix A.5), which indicates a fairly strong correlation. In the star state, interestingly, the center and any single leaf had no mutual information, yet leaves among themselves had 0.8 bits between each pair (Derived from Appendix A.5). This implies the center's state is uncorrelated when observing any one leaf alone, but the center does mediate multi-qubit correlations among leaves.

The degree vs. correlation relationship we quantified (Table 2) demonstrates that higher-degree nodes which participate in more CZ gate interactions become more correlated with the rest of the system overall. The degree-4 node in the star shares about twice as much total correlation (0.54 bits on average with each other qubit) as a degree-1 node does (0.27 bits). This suggests that degree can be a predictor of how "informed" a qubit is about the global state. However this increased correlation does not manifest as entanglement between high-degree nodes and their neighbors. Instead the high-degree node is entangled with the group as a whole. In other words the entanglement spread is delocalized: a high-degree central qubit does not form entangled pairs, but it is part of an entangled web that involves many qubits at once. This is precisely the kind of entanglement structure exploited in cluster states for one-way quantum computing, where measurement of one qubit affects the state of many others through entanglement links.

It is worth noting a subtle point, namely that mutual information is symmetric and counts all correlations, whereas concurrence/negativity specifically quantify quantum entanglement. The fact that we found $I > 0$ but $C = \mathcal{N} = 0$ for many pairs means those correlations are purely classical (in the sense of separable states). This could be verified by checking that each two-qubit ρ_{ij} can be written as a mixture of product states. Thus while the cluster state as a whole is entangled, any attempt to isolate two qubits yields at most classical correlation. This reflects the monogamy constraint quantitatively; the tangle (concurrence-squared) of a qubit with the rest equals the sum of tangles with each other qubit (Concurrence... Wikipedia). In our case, since each pair tangle is zero, all the entanglement tangle resides in higher-order (3-,4-,5-body) correlations.

From a quantum networking perspective these results indicate that simply looking at degree is not sufficient to guarantee entangled links, and that one must consider the *structure* of the state. For instance one might guess that the star center is a good entanglement distributor due to its four connections, but in the cluster state that entanglement is not in two-qubit form. However, if one were to perform local measurements on some qubits, the remaining qubits of a cluster state can be projected into entangled pairs. For example measuring three of the leaves in the star cluster (in appropriate bases) can leave the center and the remaining leaf in a Bell state. Thus the entanglement is there in potential form but not directly accessible without further operations. By measuring certain qubits, one can create entanglement between others that had none before, effectively re-routing the entanglement through the graph. Our analysis here considered the unmeasured states; including measurements would take us into the realm of entanglement percolation and purification, which is beyond our current scope.

In summary, node degree does correlate with the spread of correlations, with higher degree corresponding to more mutual information broadly, but not with two-qubit entanglement, which is uniformly zero in these cluster states. The entire 5-qubit cluster states are highly entangled as 5-partite systems even though every two-qubit subsystem is separable. This reinforces that entanglement in graph states is a global property not attributable to any single edge or pair – a feature that is crucial for their utility in quantum computing. Our expanded calculations, with explicit circuits, statevectors, partial traces, and eigenvalue analyses, provide a pedagogical verification of these concepts on a small scale.

In future work, it would be interesting to examine larger graph states and quantify at what size or graph topology two-qubit entanglement might start to appear, or to explore entanglement clustering measures that capture multi-qubit entanglement directly. Finally, our results highlight the importance of using multiple measures to characterise quantum states. If one looked only at concurrence, one might falsely conclude these cluster states carry "no entanglement" (since all $C = 0$). But through mutual information and known context we recognise they are in fact maximally entangled in a multipartite sense. This underscores that entanglement spread in a multi-qubit state cannot always be understood by pairwise metrics alone – one needs genuine multipartite metrics or a combination of indicators to fully describe it.

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A Appendix: Qiskit Code for Simulation and Analysis

Below we provide the Python code (using Qiskit 0.34 and NumPy) that was used to generate the results in this paper.

A.1 Graph State Preparation

```python
from qiskit import QuantumCircuit
from qiskit.quantum_info import Statevector

def prepare_graph_state(n, edges):
    qc = QuantumCircuit(n)
    [qc.h(q) for q in range(n)]
    [qc.cz(i, j) for i, j in edges]
    return Statevector.from_instruction(qc)

edges = {
    "path":  [(0,1), (1,2), (2,3), (3,4)],
    "cycle": [(0,1), (1,2), (2,3), (3,4), (4,0)],
    "star":  [(0,1), (0,2), (0,3), (0,4)]
}

state_path  = prepare_graph_state(5, edges["path"])
state_cycle = prepare_graph_state(5, edges["cycle"])
state_star  = prepare_graph_state(5, edges["star"])
```

A.2 Partial Trace of Two-Qubit Subsystems

```
import numpy as np
from qiskit.quantum_info import DensityMatrix, partial_trace

def two_qubit_reduced_states(state):
    rho = DensityMatrix(state)
    return {(i,j): partial_trace(rho, [k for k in range(5) if k not in
        (i,j)])
            for i in range(5) for j in range(i+1, 5)}

rhos_path  = two_qubit_reduced_states(state_path)
rhos_cycle = two_qubit_reduced_states(state_cycle)
rhos_star  = two_qubit_reduced_states(state_star)
```

A.3 Computing Entropy, Mutual Information, Concurrence, and Negativity

```
from qiskit.quantum_info import concurrence, negativity,
    mutual_information

results = {}
states = {"path": rhos_path, "cycle": rhos_cycle, "star": rhos_star}

for name, pairs in states.items():
    results[name] = {}
    for (i,j), rho in pairs.items():
        results[name][(i,j)] = {
            "C": float(concurrence(rho)),
            "N": float(negativity(rho, qargs=[0])),
            "I": float(mutual_information(rho, base=2))
        }

for pair in [(0,1), (0,2), (1,2)]:
    if pair in results["path"]:
        print(f"Pair {pair}:")
        for name in results:
            vals = results[name][pair]
            print(f"  {name.capitalize()}: C={vals['C']:.3f}, N={vals['
                N']:.3f}, I={vals['I']:.3f}")
```

A.4 Aggregating Data: Correlation Matrices and Degree Analysis

```
matrices = {name: {m: np.zeros((5,5)) for m in "CIN"} for name in
    results}

for name, pair_data in results.items():
    for (i,j), vals in pair_data.items():
        for key in "CIN":
            val = max(0.0, vals[key])
            matrices[name][key][i,j] = matrices[name][key][j,i] = val

degrees = {"path": [1,2,2,2,1], "cycle": [2]*5, "star": [4]+[1]*4}
deg_data, avg_by_degree = [], {}

for name in degrees:
```

```
    for q in range(5):
        deg = degrees[name][q]
        others = [x for x in range(5) if x != q]
        c, i, n = [np.mean(matrices[name][key][q, others]) for key in "CIN"]
        deg_data.append((deg, c, i, n, name, q))
        avg_by_degree.setdefault(deg, {"C": [], "I": [], "N": []})
        avg_by_degree[deg]["C"].append(c)
        avg_by_degree[deg]["I"].append(i)
        avg_by_degree[deg]["N"].append(n)

deg_data.sort()

print("\nDegree vs Avg Measures:")
for d, c, i, n, name, q in deg_data:
    print(f"Degree {d} ({name}, q{q}): C={c:.3f}, I={i:.3f}, N={n:.3f}")

print("\nOverall Averages per Degree:")
for d in sorted(avg_by_degree):
    ac = np.mean(avg_by_degree[d]["C"])
    ai = np.mean(avg_by_degree[d]["I"])
    an = np.mean(avg_by_degree[d]["N"])
    print(f"Degree {d}: Avg C={ac:.3f}, Avg I={ai:.3f}, Avg N={an:.3f}")
```

A.5 Table of Results

Node Degree	State	Qubit	Avg C	Avg I (bits)	Avg N
1	Path	0	0.000	0.250	0.000
1	Path	4	0.000	0.250	0.000
1	Star	1	0.000	1.000	0.000
1	Star	2	0.000	1.000	0.000
1	Star	3	0.000	1.000	0.000
1	Star	4	0.000	1.000	0.000
2	Path	1	0.000	0.250	0.000
2	Path	2	0.000	0.000	0.000
2	Path	3	0.000	0.250	0.000
2	Cycle	0	0.000	0.000	0.000
2	Cycle	1	0.000	0.000	0.000
2	Cycle	2	0.000	0.000	0.000
2	Cycle	3	0.000	0.000	0.000
2	Cycle	4	0.000	0.000	0.000
4	Star	0	0.000	1.000	0.000